Schwarzschild radius from Monte Carlo calculation of the Wilson loop in supersymmetric matrix quantum mechanics

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In the string/gauge duality it is important to understand how the space-time geometry is encoded in gauge theory observables. We address this issue in the case of the D0-brane system at finite temperature T. Based on the duality, the temporal Wilson loop operator W in gauge theory is expected to contain the information of the Schwarzschild radius $R_{\rm Sch}$ of the dual black hole geometry as $\log\langle W\rangle = R_{\rm Sch}/(2\pi\alpha' T)$. This translates to the power-law behavior $\log\langle W\rangle = 1.89\cdot(T/\lambda^{1/3})^{-3/5}$, where λ is the 't Hooft coupling constant. We calculate the Wilson loop on the gauge theory side in the strongly coupled regime by performing Monte Carlo simulation of supersymmetric matrix quantum mechanics with 16 supercharges. The results reproduce the expected power-law behavior up to a constant shift, which is explainable as α' corrections on the gravity side.

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Introduction.— String/gauge duality, which originated from the AdS/CFT correspondence [1], has been investigated intensively over the past decade. Remarkable developments that have been achieved include generalization to various cases, confirmation by explicit calculations, and applications to various branches of physics such as hadron physics and condensed matter physics.

From the viewpoint of string theory, the duality enables us to study quantum aspects of gravity including its non-perturbative effects from the gauge theory side, which is more tractable. In this regard it is important to understand how gauge theory captures the information of space-time geometry [2]. Based on the duality at finite temperature [3], one can show that a temporal Wilson loop operator in gauge theory is related directly to the Schwarzschild radius [4], which is a fundamental quantity that characterizes the dual black hole geometry. (See also Refs. [5] for related works.) In this Letter we confirm this prediction by first-principle calculations on the gauge theory side. Note that this is the first confirmation of the prescription [6] for calculating the Wilson loop based on the string/gauge duality in a non-conformal theory without protection by supersymmetry. As such, we consider our results to have impact also in applications of the duality to realistic gauge theories.

The string/gauge duality we study is the one [7] associated with a stack of N D0-branes in type IIA superstring theory at finite temperature T. The worldvolume theory of the D0-branes is given by 1d U(N) gauge theory or matrix quantum mechanics (MQM) with 16 supercharges, and the dual geometry is given by the near-extremal black 0-brane solution in type IIA supergravity. The supersymmetric MQM has been studied by Monte Carlo simulation [8] in the Fourier space [9]. In particular, the results for the internal energy at various (effective) 't Hooft coupling constant interpolated nicely the weak coupling behavior obtained by the high temperature expansion [10], and the strong coupling behavior predicted by the black hole thermodynamics of the dual geometry. (Consistent results were obtained also by using a lattice approach [11].)

Here we apply this method to the calculation of the temporal Wilson loop operator, and demonstrate that one can extract the Schwarzschild radius of the dual black hole geometry from it. See Ref. [12] for earlier discussions on a similar issue in the same model using other observables and other calculation techniques.

Wilson loop in the dual string theory.— Let us review the calculation of the Wilson loop based on the string/gauge duality [6] for general D-branes. In addition to a stack of N D-branes, which are placed on top of each other creating a curved background geometry, we consider a single probe D-brane, which is placed far away from them in parallel. Since D-branes are objects which a fundamental string can end on, we may consider such a string stretched between the probe D-brane and one of the N D-branes. The amplitude for the string propagating along a certain loop $\mathcal{C}$ on the D-brane can be calculated in two different ways.

First in the worldvolume theory of the N D-branes, the process is viewed as a heavy test particle in the fundamental representation of the U(N) group propagating along the loop. The amplitude is therefore given by

$$\mathcal{A} = \langle W(\mathcal{C})\rangle\, \mathrm{e}^{-M\ell}\ , \tag{1}$$

where $W(\mathcal{C})$ represents the Wilson loop associated with

the loop $\mathcal{C}$, whose perimeter has the length ℓ. The mass M of the test particle, which appears in (1), is given by the distance of the N D-branes and the probe D-brane.

Next we view the same process on the gravity side as a string propagating in the curved space-time background, which is created by the N D-branes. The amplitude is calculated by the path-integral over the worldsheet attached to the loop $\mathcal{C}$ on the probe D-brane as

$$\mathcal{A} = \frac{1}{N} \int_{\mathcal{C}} \mathrm{e}^{-S_{\rm string}} \ , \tag{2}$$

where $S_{\rm string}$ represents the worldsheet action, whose bosonic part is given by the Polyakov action

$$S_{\rm P} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \Big(h^{ab} g_{MN} \partial_a x^M \partial_b x^N + \alpha' \phi R_{(2)} \Big) \ . \tag{3}$$

Here x^M represents the embedding of the worldsheet into the target space with the metric g_{MN} $(M, N = 1, \ldots 10)$, while $R_{(2)}$ represents the two-dimensional scalar curvature defined for the worldsheet metric h_{ab} $(a, b = 1, 2)$. The effective string coupling is given as e^{ϕ} in terms of the dilaton field ϕ. We have omitted a term in (3) depending on the NS-NS B-field since the background we are considering does not have non-zero B field.

In what follows, we will be mostly interested in the parameter region, in which the string coupling is so small that we only have to consider the disk amplitude in (2). We also restrict ourselves to small α', which corresponds to a large string tension, so that the path integral (2) is dominated by the saddle-point configuration. In this parameter region, one can use the classical solution to the supergravity as the background. According to the dictionary of the string/gauge duality, the parameter region corresponds to taking the planar large-N limit with large 't Hooft coupling constant on the gauge theory side.

Equating (1) and (2), we obtain a formula which relates the Wilson loop in the strongly coupled gauge theory to the string amplitude on the classical background geometry. The explicit check of this formula has been discussed only in highly symmetric cases. In the 4d $\mathcal{N} = 4$ super Yang-Mills theory (SYM), in particular, it is argued that the gauge theory computation of a circular Wilson loop, which is half BPS, reduces to a matrix integration with a Gaussian weight [13, 14, 15]. The obtained result indeed agrees with the prediction from the gravity side. The agreement can be understood also from the scale invariance of the worldsheet theory near the D3-branes [16].

The D0-brane case.— From now on, let us restrict ourselves to the D0-brane case. The gauge theory side is described by the supersymmetric MQM

$$\begin{aligned} S_{\rm SQM} \ = \ & \frac{N}{\lambda} \int_0^{\beta} dt \, \mathrm{tr} \left\{ \frac{1}{2} (D_t X_i)^2 - \frac{1}{4} [X_i, X_j]^2 \right. \\ & \left. + \frac{1}{2} \psi_\alpha D_t \psi_\alpha - \frac{1}{2} \psi_\alpha (\gamma_i)_{\alpha\beta} [X_i, \psi_\beta] \right\} \ , \end{aligned} \tag{4}$$

where $D_t = \partial_t - i\,[A(t), \ \cdot \ ]$ represents the covariant derivative with the gauge field $A(t)$ being an $N \times N$ Hermitian matrix. The model can be viewed as a 1d U(N) gauge theory with adjoint matters $X_i(t)$ $(i = 1, \cdots, 9)$ and $\psi_\alpha(t)$ $(\alpha = 1, \cdots, 16)$, which are bosonic and fermionic matrices, respectively. The extent of the Euclidean time direction β corresponds to the inverse temperature $\beta \equiv T^{-1}$, and the fermions ψ_α obey anti-periodic boundary conditions.

We consider the loop $\mathcal{C}$ to be winding once around the temporal direction. The corresponding Wilson loop operator in the gauge theory is given by [6]

$$W = \frac{1}{N} \, \mathrm{tr} \, \mathrm{Pexp} \int_0^{\beta} dt \Big(iA(t) + n_i X_i(t) \Big) \ , \tag{5}$$

where $\vec{n}$ is a unit vector in R^9 specifying the direction in which the probe D0-brane is separated. Note that the adjoint scalar appears in (5) unlike the definition of the Polyakov line since the end of the string is coupled not only to the gauge field but also to the adjoint scalar. The overall factor $1/N$ is introduced to make the quantity finite in the planar large-N limit. We have used the same normalization on the right hand side of (2).

The gravity dual of the supersymmetric MQM is given by the near-horizon geometry of the (Euclidean) near-extremal black 0-brane solution in type IIA supergravity. In particular, the metric is given by [7]

$$\frac{ds^2}{\alpha'} = \frac{U^{7/2} f(U)}{\sqrt{d_0 \lambda}} dt^2 + \frac{\sqrt{d_0 \lambda}}{U^{7/2} f(U)} dU^2 + \frac{\sqrt{d_0 \lambda}}{U^{3/2}} d\Omega_8^2 \ , \tag{6}$$

where $f(U) = 1 - U_0^7/U^7$ and $d_0 \equiv 2^7 \pi^{9/2} \Gamma(7/2)$. The Schwarzschild radius and the inverse Hawking temperature are given by

$$R_{\rm Sch} = \alpha' U_0 \, , \qquad \beta = \frac{4}{7} \pi \sqrt{d_0 \lambda} U_0^{-5/2} \ . \tag{7}$$

Let us evaluate the string disk-amplitude (2) in the background geometry (6). In the $\alpha' \to 0$ limit, the second term in (3) can be omitted, and one can replace the string action by the Nambu-Goto action $S_{\rm NG}$, which is nothing but the area of the string worldsheet times the string tension. Following the proposal [6], we consider a string worldsheet localized in the S^8 direction. Then, due to the form of the metric (6), the Nambu-Goto action for the minimal area is given by $S_{\rm NG} = \frac{1}{2\pi} \beta (U_\infty - U_0)$, where U_∞ represents the position of the probe D0-brane.

Since the perimeter ℓ of the Wilson loop in (1) is given by $\ell = \beta$ in the present set-up, we obtain the identity

$$\log \langle W(\mathcal{C}) \rangle - \beta M = \frac{\beta U_0}{2\pi} - \frac{\beta U_\infty}{2\pi} \ . \tag{8}$$

Considering that the mass of the test particle on the gauge theory side is given by the position of the probe D0-brane, it is natural to identify the second terms on

both sides of (8). This follows also from the prescription proposed in Ref. [17] based on T-duality. Thus we obtain

$$\log\langle W(\mathcal{C})\rangle = \frac{\beta U_0}{2\pi} = \frac{\beta R_{\rm Sch}}{2\pi\alpha'} = 1.89\left(\frac{T}{\lambda^{1/3}}\right)^{-3/5} , \quad (9)$$

where we have used (7).

The range of validity.— Let us recall the range of validity for the supergravity description [7]. By changing the target-space coordinates as $U = U_0 u^{2/5}$ and $t = \frac{2}{5}\sqrt{d_0\lambda}U_0^{-5/2}\tau$, the metric (6) and the effective string coupling e^{ϕ} become

$$\frac{ds^2}{\alpha'} = (d_0^{1/3}\mathcal{K})^{3/5}\left[\frac{4}{25}\left(\tilde{f}(u)d\tau^2 + \frac{du^2}{\tilde{f}(u)}\right) + d\Omega_8^2\right], \quad (10)$$

$$\mathrm{e}^{\phi} = \frac{(2\pi)^2}{N}(d_0^{-1/7}\mathcal{K})^{21/10} , \qquad \mathcal{K} = \frac{7\lambda^{1/3}}{4\pi u T} , \quad (11)$$

where $\tilde{f}(u) \equiv u^2(1 - u^{-14/5})$. From (10), one finds that the geometry asymptotes at large u to a geometry which is conformally equivalent to $\mathrm{AdS}_2\times\mathrm{S}^8$ [18], and that the typical length scale of the geometry is given by $\rho \equiv (uT/\lambda^{1/3})^{-3/10}\alpha'^{1/2}$. This scale should be much larger than the string length $\alpha'^{1/2}$ for the α' corrections to the supergravity action to be negligible. Hence, $uT/\lambda^{1/3} \ll 1$. In this case, the first term in (3), which is proportional to ρ^2, becomes large, and the semi-classical treatment for the string amplitude (2) is also justified. Note, however, that we have introduced U_∞. Assuming that we only need to require U_∞/U_0 to be large (but finite), we may assume u to be finite as well. Then we obtain the condition $T/\lambda^{1/3} \ll 1$.

We also need to require the effective string coupling e^{ϕ} to be small. From (11), we obtain $N^{-10/21} \ll T/\lambda^{1/3}$ noting that $u \geq 1$ in our finite temperature set-up.

α' corrections.— Let us discuss possible subleading terms in (9) due to α' corrections on the gravity side. There are three effects one should consider: (I) the coupling with the background ϕ field represented by the second term in (3), (II) α'-corrections to the background fields that appear in the action (3), and (III) the quantum fluctuation of the string worldsheet including fermionic degrees of freedom in evaluating (2). In order to discuss the next-leading order terms, we can treat each of these effects separately.

The effect (I) yields a constant term and a logarithmic term with respect to $T/\lambda^{1/3}$ in (9) as one can see from (3) and (11). The constant term includes $\log N$, but this is canceled by the prefactor $1/N$ in (2) as it should. The effect (II) can be neglected at this order since α'-corrections to the type IIA supergravity action starts only at the α'^3 order [19]. The effect (III) yields a constant term to (9). This effect is discussed also in the case of D3-branes [20]. In fact a logarithmic term can appear from it as well due to the insertion of the ghost zero mode [14].

Monte Carlo simulation.— We perform Monte Carlo simulation of the model (4) and calculate the temporal Wilson loop (5) to check the prediction (9). We use the Fourier-mode simulation method [9], in which we take the static diagonal gauge $A(t) = \frac{1}{\beta}\mathrm{diag}(\alpha_1, \cdots, \alpha_N)$ with $-\pi < \alpha_a \leq \pi$, and introduce a cutoff Λ on the Fourier modes as $X_i(t) = \sum_{n=-\Lambda}^{\Lambda}\tilde{X}_{in}\mathrm{e}^{i\omega n t}$, where $\omega = \frac{2\pi}{\beta}$. Supersymmetry at $T = 0$, which is broken only due to finite Λ, is shown to recover rapidly as $\Lambda \to \infty$ in a simpler model [9]. The effective 't Hooft coupling constant is given by $\lambda_{\rm eff} = \lambda/T^3$. In actual simulation we set $\lambda = 1$ without loss of generality, so that high/low T corresponds to weak/strong coupling, respectively.

Integration over the fermionic matrices yields a Pfaffian $\mathrm{Pf}\mathcal{M}$, which is complex in general. According to the standard reweighting method, one uses $|\mathrm{Pf}\mathcal{M}|$ to generate configurations, and includes the effect of the phase when one calculates the expectation values. In fact $\mathrm{Pf}\mathcal{M}$ is almost real positive at sufficiently high T, but the fluctuation of the phase becomes larger as T decreases, which causes the so-called sign problem. It turned out, however, that the results of the reweighting method in the temperature regime where the sign problem is not so severe are actually in good agreement with what we obtain by simply neglecting the phase. We interpret this as an effect of the large-N limit, in which the fluctuations of single trace observables vanish. For the same reason, it is expected that $\log\langle|W|\rangle$ agrees with $\langle\log|W|\rangle$ in the large-N limit. We therefore calculate the latter in an ensemble generated with $|\mathrm{Pf}\mathcal{M}|$. Complete justification of these simplifications is left for future investigations.

We evaluate (5) as a limit $W = \lim_{\nu\to\infty} W_\nu$, where

$$W_\nu = \frac{1}{N}\mathrm{tr}\prod_{k=0}^{\nu-1}\left[1 + \frac{\beta}{\nu}\Big\{iA + n_i X_i(t_k)\Big\}\right] \quad (12)$$

with $t_k = \frac{k}{\nu}\beta$. The matrices $X_i(t_k)$ are obtained as the inverse Fourier transform of the configurations generated by our simulation. Using the asymptotic behavior $W_\nu \simeq W + \frac{\mathrm{const.}}{\nu}$ at large ν, we can make a reliable extrapolation to $\nu = \infty$. As the unit vector $\vec{n}$, we have used the ones in all 9 directions with plus or minus sign in front, and averaged over them to increase statistics.

In Fig. 1 we plot $\langle\log|W|\rangle$ against $T^{-3/5}$. As T decreases (to the right on the horizontal axis), the data show a clear linear growth with a slope consistent with the value 1.89 predicted in (9). In fact we can fit our data to $\langle\log|W|\rangle = 1.89T^{-3/5} - C$, where $C = 4.95$ for $N = 4$ and $C = 4.58$ for $N = 6$. The data points for $N = 8$ are very close to those for $N = 6$. Note that the constant term and the logarithmic term predicted from the gravity side are difficult to distinguish numerically. We therefore consider that the value of C extracted above actually represents the sum of the two terms at the temperature regime investigated.

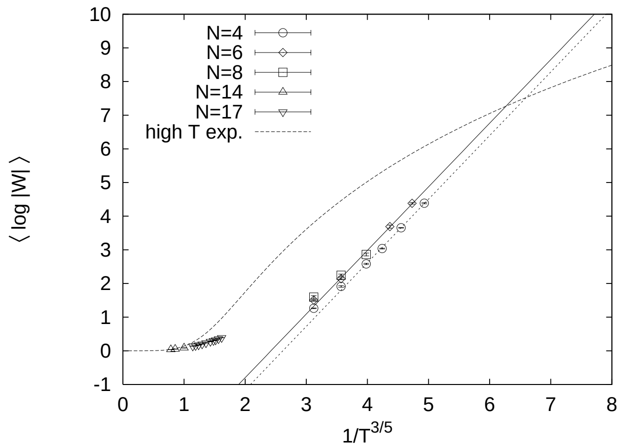

FIG. 1: The plot of $\langle \log |W| \rangle$ for $\lambda = 1$ against $T^{-3/5}$. The cutoff Λ is chosen as follows: $\Lambda = 12$ for $N = 4$; $\Lambda = 0.6/T$ for $N = 6, 8$; $\Lambda = 4$ for $N = 14$; $\Lambda = 6$ for $N = 17$. The dashed line represents the results of the high-temperature expansion up to the next-leading order with extrapolations to $N = \infty$, which are obtained by applying the method in Ref. [10]. The solid line and the dotted line represent fits for $N = 6$ and $N = 4$ respectively, to straight lines with the slope 1.89 predicted from the gravity side at the leading order.

Summary.— We have presented the first Monte Carlo calculations of the Wilson loop in a supersymmetric gauge theory at strong coupling. Up to subleading terms anticipated from the analysis on the gravity side, our results are in precise agreement with the prediction from the dual supergravity. This is a new and highly nontrivial evidence for the string/gauge duality. It would be nice to obtain the subleading terms explicitly from the gravity side, which will provide a nontrivial check of the duality including α' corrections. It is also interesting to extend this work to $\mathcal{N} = 4$ SYM on $\mathrm{R} \times \mathrm{S}^3$, which is possible by using the equivalence [21] in the planar limit between the SYM and a mass-deformed MQM around a multi-fuzzy-sphere background. The equivalence is confirmed by explicit calculations at weak coupling [22].

The fact that we were able to see the Schwarzschild radius of the dual black hole geometry by simulating large-N matrices gives us strong support and a firm ground for using matrix model simulations to study quantum gravity [23]. Note that the gauge theory description is valid also at small λ and small N, where the dual supergravity description is no longer valid. Of particular interest is to study the parameter region corresponding to M-theory.

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[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).
[2] D. Berenstein, JHEP **0601**, 125 (2006).
[3] E. Witten, Adv. Theor. Math. Phys. **2**, 505 (1998).
[4] S. J. Rey, S. Theisen and J. T. Yee, Nucl. Phys. B **527**, 171 (1998); M. Kruczenski and A. Lawrence, JHEP **0607**, 031 (2006); M. Headrick, Phys. Rev. D **77**, 105017 (2008).
[5] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, Phys. Lett. B **434**, 36 (1998); JHEP **9806**, 001 (1998).
[6] S. J. Rey and J. T. Yee, Eur. Phys. J. C **22**, 379 (2001); J. M. Maldacena, Phys. Rev. Lett. **80**, 4859 (1998).
[7] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, Phys. Rev. D **58**, 046004 (1998).
[8] K. N. Anagnostopoulos, M. Hanada, J. Nishimura and S. Takeuchi, Phys. Rev. Lett. **100**, 021601 (2008).
[9] M. Hanada, J. Nishimura and S. Takeuchi, Phys. Rev. Lett. **99**, 161602 (2007).
[10] N. Kawahara, J. Nishimura and S. Takeuchi, JHEP **0712**, 103 (2007).
[11] S. Catterall and T. Wiseman, Phys. Rev. D **78**, 041502 (2008); JHEP **0712**, 104 (2007).
[12] D. Kabat, G. Lifschytz and D. A. Lowe, Phys. Rev. D **64**, 124015 (2001).
[13] J. K. Erickson, G. W. Semenoff and K. Zarembo, Nucl. Phys. B **582**, 155 (2000).
[14] N. Drukker and D. J. Gross, J. Math. Phys. **42**, 2896 (2001).
[15] V. Pestun, arXiv:0712.2824.
[16] H. Kawai and T. Suyama, Nucl. Phys. B **789**, 209 (2008).
[17] N. Drukker, D. J. Gross and H. Ooguri, Phys. Rev. D **60**, 125006 (1999).
[18] A. Jevicki and T. Yoneya, Nucl. Phys. B **535**, 335 (1998); A. Jevicki, Y. Kazama and T. Yoneya, Phys. Rev. D **59**, 066001 (1999); Y. Sekino and T. Yoneya, Nucl. Phys. B **570**, 174 (2000).
[19] D.J. Gross and E. Witten, Nucl. Phys. B **277**, 1 (1986).
[20] J. Greensite and P. Olesen, JHEP **9904**, 001 (1999); S. Forste, D. Ghoshal and S. Theisen, JHEP **9908**, 013 (1999); S. Naik, Phys. Lett. B **464**, 73 (1999); Y. Kinar, E. Schreiber, J. Sonnenschein and N. Weiss, Nucl. Phys. B **583**, 76 (2000); N. Drukker, D. J. Gross and A. A. Tseytlin, JHEP **0004**, 021 (2000); M. Kruczenski and A. Tirziu, JHEP **0805**, 064 (2008).
[21] T. Ishii, G. Ishiki, S. Shimasaki and A. Tsuchiya, Phys. Rev. D **78**, 106001 (2008).
[22] G. Ishiki, S. W. Kim, J. Nishimura and A. Tsuchiya, arXiv:0810.2884; Y. Kitazawa and K. Matsumoto, arXiv:0811.0529.
[23] D. Berenstein and R. Cotta, JHEP **0704**, 071 (2007); K. N. Anagnostopoulos and J. Nishimura, Phys. Rev. D **66**, 106008 (2002); J. Ambjørn et al., JHEP **0007**, 011 (2000).